Homework 1
Algebraic Topology

Joshua Ruiter

February 12, 2018

Note: When unspecified, a map is assumed to be continuous.

Lemma 0.1 (not assigned, just stated for clarity)**.** *Composition of continuous maps is continuous.*

Definition 0.1. Let X, Y be spaces and $f : X \to Y$ be a continuous map. Consider the space $(X \times I) \sqcup Y$, and define an equivalence relation $(x, 1) \sim f(x)$. Then we define the **mapping cylinder** of f, denoted M_f, to be $(X \times I) \sqcup Y / \sim$.

Definition 0.2. Let X be a space and $\sim$ an equivalence relation on X. The **quotient space** $X/\sim$ is the set of equivalence classes,

$$(X/\sim) = \{[x] : x \in X\}$$

Note that $\pi : X \to X/\sim$ given by $x \mapsto [x]$ is surjective. We define a set $U \subset X/\sim$ to be open if $\pi^{-1}(U)$ is open in X. This gives rise to a topology on $X/\sim$, called the **quotient topology**.

Definition 0.3. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Let X be the cartesian set product $\prod_{i\in I} X_i$. We define the **product topology** on X by defining open sets to be sets of the form $\prod_{i\in I} U_i$ where $U_i \subset X_i$ is open and $U_i \neq X_i$ for only finitely many i.

Definition 0.4. A **CW comlex** or **cell complex** is a space built up from ataching n-cells to $n-1$ cells. More precisely, begin with a set X^0 of points (0-cells). Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\phi_\alpha : S^{n-1} \to X^{n-1}$. That is, X^n is the space

$$X^n = \left(X^{n-1} \bigsqcup_\alpha D^n_\alpha \right) / \sim$$

where $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha \cong S^{n-1}$. That is,

$$X^n = X^{n-1} \bigsqcup_\alpha e^n_\alpha$$

If this process terminates for some n, then $X = X^n$ has the expected quotient topology. If the process does not terminate, then $A \subset X$ is open iff $A \cap X^n$ is open in X^n for every n. (This is called the **weak topology**.)

Definition 0.5. Let C, D be categories, and let $F, G : C \to D$ be covariant functors. A **natural transformation** $\eta : F \to G$ is assigns each object $X \in \mathrm{Ob}(C)$ to a morphism $\eta_X : F(X) \to G(X)$ such that for every morphism $f : X \to Y$, we have $\eta_Y \circ F(f) = G(f) \circ \eta_X$. That is, the diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta_X}\downarrow & & {\scriptstyle \eta_Y}\downarrow \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

Definition 0.6. Let X be a topological space and $A \subset X$. The pair (X, A) has the **homotopy extension property** if for every homotopy $f_t : A \to Y$ and every map $F_0 : X \to Y$ such that $F_0|_A = f_0$, there exists a homotopy $F_t : X \to Y$ such that $F_t|_A = f_t$ for all t.

(Exercise 2)
We construct an explicit deformation retraction of $\mathbb{R}^n \setminus \{0\}$ onto $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Define $f_t : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ by

$$f_t(x) = (1-t)x + t\frac{x}{|x|}$$

Then $f_0(x) = x$ so $f_0 = \mathrm{Id}_{\mathbb{R}^n \setminus \{0\}}$. Also, $f_1(x) = \frac{x}{|x|} \in S^{n-1}$, which is surjective onto the circle, so $f_1(\mathbb{R}^{n-1} \setminus \{0\}) = S^{n-1}$. Finally, $f_t|_{S^{n-1}} = \mathrm{Id}_{S^{n-1}}$ because if $x \in S^{n-1}$, then $|x| = 1$ so $f_t(x) = (1-t)x + tx = x$. Thus f_t is the required deformation retraction.

The next two lemmas say that we can right- and left-compose with homotopic maps to get homotopic maps. These provide a very clean proof that the composition of homotopy equivalences is a homotopy equivalence.

Lemma 0.2 (for Exercise 3a)**.** *Let $f_0, f_1 : X \to Y$ be homotopic and $g : Y \to Z$. Then $gf_0 \simeq gf_1$.*

Proof. Let $f_t : X \to Y$ be a homotopy from f_0 to f_1. Then $gf_t : X \to Y$ is a homotopy from gf_0 to gf_1. □

Lemma 0.3 (for Exercise 3a)**.** *Let $f_0, f_1 : X \to Y$ be homotopic and $h : Z \to X$. Then $f_0 h \simeq f_1 h$.*

Proof. Let $f_t : X \to Y$ be a homotopy from f_0 to f_1. Then $f_t h$ is a homotopy from $f_0 h$ to $f_1 h$. □

Proposition 0.4 (Exercise 3a)**.** *A composition of homotopy equivalences is a homotopy equivalence. Thus homotopy equivalence is an equivalence relation.*

Proof. Let $f_1 : X \to Y$ and $f_2 : Y \to Z$ be homotopy equivalences. We will show that the composition $f_2 f_1$ is a homotopy equivalence.

Since f_1, f_2 are homotopy equivalences, there exist maps $g_1 : Y \to X$ and $g_2 : Z \to Y$ such that $f_1 g_1 \simeq \mathrm{Id}_Y, g_1 f_1 \simeq \mathrm{Id}_X, f_2 g_2 \simeq \mathrm{Id}_Z$, and $g_2 f_2 \simeq \mathrm{Id}_Y$. Then since $f_1 g_1 \simeq \mathrm{Id}_Y$, by the Lemma 0.2 above, $f_2 f_1 g_1 \simeq f_2 \mathrm{Id}_Y = f_2$. Then by Lemma 0.3, $f_2 f_1 g_1 g_2 \simeq f_2 g_2 \simeq \mathrm{Id}_Z$. Similarly,

$$g_2 f_2 \simeq \mathrm{Id}_Y \implies g_1 g_2 f_2 \simeq g_1 \mathrm{Id}_Y = g_1 \implies g_1 g_2 f_2 f_1 \simeq g_1 f_1 \simeq \mathrm{Id}_X$$

Thus we have $(f_2 f_1)(g_1 g_2) \simeq \mathrm{Id}_Z$ and $(g_1 g_2)(f_2 f_1) \simeq \mathrm{Id}_X$. Thus $g_1 g_2$ is the required homotopy inverse for $f_1 f_2$, hence $f_1 f_2$ is a homotopy equivalence between X and Z. This establishes transitivity of homotopy equivalence.

Reflexivity and symmetric are quick to show. (Reflexivity) Let X be a space. Then Id_X is a homotopy equivalence from X to itself, since there exists a map, namely Id_X, such that $\mathrm{Id}_X \mathrm{Id}_X \simeq \mathrm{Id}_X$. Thus $X \simeq X$. (Symmetry) Let $X \simeq Y$ via f. Then the map $g : Y \to X$ where $fg \simeq \mathrm{Id}_Y$ and $gf \simeq \mathrm{Id}_X$ is a homotopy equivalence from Y to X, so $Y \simeq X$. □

Proposition 0.5 (Exercise 3b)**.** *Let X, Y be spaces. The $\simeq$ is an equivalence relation on maps $f : X \to Y$.*

Proof. Let $f, g, h : X \to Y$ be continuous. Then $f \simeq f$ by the homotopy $f_t(x) = f(x)$. If $f \simeq g$ via a homotopy f_t, then $g \simeq f$ via f_{1-t}. Finally, suppose $f \simeq g$ via f_t and $g \simeq h$ via g_t. Then define

$$\phi_t(x) = \begin{cases} f_{2t}(x) & 0 \le t \le 1/2 \\ g_{2t-1}(x) & 1/2 \le t \le 1 \end{cases}$$

Note that $\phi_{1/2}(x)$ is well defined because $f_1(x) = g(x) = g_0(x)$. Also note that ϕ is continuous by the Gluing Lemma. Then ϕ is a homotopy between f and h, so $f \simeq h$. □

Proposition 0.6 (Exercise 3c)**.** *Let $f_0, f_1 : X \to Y$ such that f_0 is a homotopy equivalence and $f_0 \simeq f_1$. Then f_1 is a homotopy equivalence.*

Proof. Since f_0 is a homotopy equivalence, there exists a map $g : Y \to X$ such that $f_0 g \simeq \mathrm{Id}_Y$ and $g f_0 \simeq \mathrm{Id}_X$. Then since $f_0 \simeq f_1$, by Lemma 0.2, $g f_0 \simeq g f_1$ and since $\simeq$ is transitive, $g f_0 \simeq \mathrm{Id}_X \simeq g f_1$. Likewise, since $f_0 \simeq f_1$ by Lemma 0.3 $f_0 g \simeq f_1 g$ so $f_1 g \simeq \mathrm{Id}_Y$. Thus f_1 is also a homotopy equivalence from X to Y via the same map g. □

Proposition 0.7 (Exercise 4)**.** *If X deformation retracts to A in the weak sense, then the inclusion $\iota : A \hookrightarrow X$ is a homotopy equivalence.*

Proof. Let $f_t : X \to X$ be a weak deformation retraction, that is, f_t is a homotopy such that $f_0 = \mathrm{Id}_X$, $f_1(X) \subset A$, and $f_t(A) \subset A$. Then we have that $f_1\iota \simeq \mathrm{Id}_A$ via the homotopy $\widetilde{f_t} = f_t|_A = f_t\iota : A \to A$, since $\widetilde{f_0} = f_0|_A = \mathrm{Id}_X|_A = \mathrm{Id}_A$, and $\widetilde{f_1} = f_1|_A = f_1\iota$. (Note that $\widetilde{f_t}(A) \subset A$ because $f_t(A) \subset A$.) We also have that $\iota f_1 \simeq \mathrm{Id}_X$, via the homotopy $f_t : X \to X$, since $f_0 = \mathrm{Id}_X$ and $f_1 = \iota f_1$ since $f_1(X) \subset A$. Thus f_1 is a homotopy inverse for ι, so ι is a homotopy equivalence. □

Proposition 0.8 (Exercise 5, not assigned, but needed for Exercise 6)**.** *If a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X there exists a neighborhood $V \subset U$ of x such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.*

Lemma 0.9 (for Exercise 6a)**.** *Let X be a space and $A \subset X$. If the inclusion $\iota_A : A \hookrightarrow X$ is nullhomotopic, then A lies in a single path component of X.*

Proof. Let $f_t : A \to X$ be a homotopy with $f_0 = \iota_A$ and $f_1(A) = x_0$. Let $y \in A$. Then define $\gamma : I \to X$ by $\gamma(t) = f_t(y)$. We have $\gamma(0) = y$ and $\gamma(1) = x_0$, so y is contained in the path component of x_0 in X. Since y was arbitrary, all of A lies in the path component of x_0. □

Lemma 0.10 (for Exercise 6)**.** *Let X be a space, and let $A \subset B \subset X$, and suppose that there is a deformation retraction $f_t : X \to X$ of X onto B and a deformation retraction $g_t : B \to B$ of B onto A. Then there is a deformation retraction of X onto A.*

Proof. Let $f_t : X \to X$ and $g_t : B \to B$ be deformation retractions, i.e.

$$\begin{aligned} &f_0 = \mathrm{Id}_X \qquad && f_1(X) = B \qquad && f_t|_B = \mathrm{Id}_B \\ &g_0 = \mathrm{Id}_B \qquad && g_1(B) = A \qquad && g_t|_A = \mathrm{Id}_A \end{aligned}$$

Define $h_t : X \to X$ by

$$h_t(x) = \begin{cases} f_{2t}(x) & 0 \le t \le 1/2 \\ g_{2t-1} \circ f_1(x) & 1/2 \le t \le 1 \end{cases}$$

For $t = 1/2$, the two alternate definitions agree, because $f_{2(1/2)} = f_1$ and $g_{2(1/2)-1} \circ f_1 = g_0 \circ f_1 = \mathrm{Id}_B \circ f_1 = f_1$. Hence h is well-defined and continuous by the Gluing Lemma. Furthermore, h is a deformation retraction of X onto A, as $h_0 = f_0 = \mathrm{Id}_X$, and $h_1(X) = g_1 \circ f_1(X) = g_1(B) = A$ and

$$\begin{aligned} h_t|_A &= \begin{cases} f_{2t}|_A & 0 \le t \le 1/2 \\ (g_{2t-1} \circ f_1)|_A & 0 \le t \le 1/2 \end{cases} \\ &= \begin{cases} \mathrm{Id}_A & 0 \le t \le 1/2 \\ g_{2t-1} \circ \mathrm{Id}_A & 0 \le t \le 1/2 \end{cases} \\ &= \begin{cases} \mathrm{Id}_A & 0 \le t \le 1/2 \\ \mathrm{Id}_A & 0 \le t \le 1/2 \end{cases} \\ &= \mathrm{Id}_A \end{aligned}$$

□

Proposition 0.11 (Exercise 6a). *Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$ for $r \in \mathbb{Q} \cap [0,1]$. Then X deformation retracts to any point in the segment $[0,1] \times \{0\}$ but not to any other point.*

Proof. First we show that X deformation retracts onto $A = [0,1] \times \{0\}$. Geometrically, we retract each of the line segments $\{r\} \times [0, 1-r]$ straight down toward the x-axis. More formally, define $f_t : X \to X$ by $f_t(x,y) = (x, (1-t)y)$. Note that this does map into X. Also, $f_0(x,y) = (x,y)$ and $f_1(x,y) = (x,0)$ and $f_t|_A = \mathrm{Id}_A$, so f_t is a deformation retraction.

Now we show that A retracts onto any point inside itself. Let $(x_0, 0) \in [0,1] \times \{0\}$. Define $g_t : A \to A$ by $g_t(x,0) = ((1-t)x + tx_0, 0)$. Then $g_0(x,0) = (x,0)$ and $g_1(x,0) = (x_0,0)$ and $g_t(x_0,0) = (x_0,0)$ so g_t is a deformation retraction of A onto $(x_0,0)$. Thus X deformation retracts onto $(x_0,0)$.

Now we show that X does not deformation retract onto any other point. Suppose X retracts onto (x,y) with $y \neq 0$. Then the ball centered at (x,y) with radius $\frac{1}{2}y$ does not intersect the x-axis. Let U be the intersection of this ball with X. Then U is a disjoint union of vertical (open) line segments, each of which constitutes a separate path component. In particular, the path component containing (x,y) is $U \cap (\{x\} \times [0, 1-x])$.

By Exercise 5, there is a neigborhood $V \subset U$ of (x,y) such that the inclusion map $V \hookrightarrow U$ is nullhomotopic, so by the above lemma, V lies in a single path component of U. Since $(x,y) \in V$, we have $V \subset U \cap (\{(x\} \times [0,1-x])$. Since V is an open neighborhood of (x,y), for some $\epsilon > 0$ we must have $B((x,y),\epsilon) \cap X \subset V$. Because the rationals are dense, for any $\epsilon > 0$, there is a rational $r \neq x$ such that $0 < r < \epsilon$, so $B((x,y),\epsilon) \cap (\{r\} \times [0,1-r]) \neq \emptyset$. Thus V must intersect another path component of U nontrivially. This is a contradiction, so we conclude that X does not deformation retract onto any point off of the x-axis. □

(I broke Exercise 6b into two separate propositions because showing that Y is contractible became a very very long proof.)

Proposition 0.12 (Exercise 6b, part one). *Let Y be the subspace of $\mathbb{R}^2$ that is the union of infinitely many copies of X (see picture in Hatcher on page 18). The Y is contractible.*

Proof. Let Z be the zigzag subspace, let $y \in Y$. We define a path $\gamma_y : [0,\infty) \to Y$ by setting $\gamma_y(t)$ to be the point in Y by "flowing" the point y along Y with velocity one toward the right. So for $y \in Z$, $\gamma_t(y)$ is the point in Z to the right of y that has distance t from y, in the sense of traveling only in Y. For a point y in one of the "comb hairs" of length a, for $t \leq a$, $\gamma_t(y)$ is the point on the same comb hair at distance $a - t$ from the base of that comb hair. At time a, we have $\gamma_a(y) \in Z$, and after that we already defined what $\gamma_t(y)$ does. Then we define $h_t : Y \to Y$ by $h_t(y) = \gamma_y(t)$. We claim that $h_1 : Y \to Z$ is a homotopy equivalence. (Note that h_1 does in fact map into Z, since no point in Y is more than distance one from Z, while flowing along Y.)

We need to show several things: that h_1 is continuous, and that there is a continuous function $g : Z \to Y$ such that $g \circ h_1 \simeq \mathrm{Id}_Y$ and $h_1 \circ g \simeq \mathrm{Id}_Z$. First, we show that h_1 is continuous. It is sufficient to show that the preimage of an open "interval" of Z is open in Y.

(Note: Y has the subspace topology from $\mathbb{R}^2$). By an interval of Z, we mean the intersection Z with an open ball in $\mathbb{R}^2$. First consider a single point $y \in Z$. The preimage $h_1^{-1}(y)$ is the intersection of a vertical line in $\mathbb{R}^2$ with Y. Thus, if U is an open interval in Z, the preimage is the union of of an interval of vertical lines in $\mathbb{R}^2$, that is, $h_1^{-1}(U)$ is the intersection of Y with an infinite open rectangle from $\mathbb{R}^2$. Thus $h_1^{-1}(U)$ is open, so h_1 is continuous.

Now we need a homotopy inverse for h_1. Define $g_t : Z \to Y$ by sending the point z to the point on Z at distance t (flowing along Y) to the left of y. Then g_t is continuous, since the preimage of any U is a "shift" of U by flowing it to the right along Z. (The preimage of any point in one of the comb hairs is empty.) We will show that g_1 is a homotopy inverse for h_1. We see immediately that $h_1 \circ g_1 = \mathrm{Id}_Z \simeq \mathrm{Id}_Z$, so all that remains is to show $g_1 \circ h_1 \simeq \mathrm{Id}_Y$.

First, we claim that $h_1 \simeq \mathrm{Id}_Y$ via h_t for $t \in [0,1]$. We have $h_0 = \mathrm{Id}_Y$ and at $t = 1$ we have h_1, so we need to show that the map $(y,t) \mapsto h_t(y)$ is continuous. Consider a single point $y_0 \in Y$. The preimage of y_0 a the "slice" of the form

$$\{(y,t) \in Y \times [0,1] : h_t(y) = y_0\} = \bigcup_{t \in [0,1]} \left(h_t^{-1}(y_0)\right) \times \{t\}$$

For an open set in Y, the preimage is a union over such slices, which is open in the product topology on $Y \times [0,1]$, so $(y,t) \mapsto h_t(y)$ is continuous. Thus $h_1 \simeq \mathrm{Id}_Y$. Now we claim that $g_1 \circ h_1 \simeq h_1$. Define $f_t : Y \to Y$ by $f_t = g_t \circ h_1$. Then

$$\begin{aligned} f_0 &= g_0 \circ h_1 = \mathrm{Id}_Z \circ h_1 = h_1 \\ f_1 &= g_1 \circ h_1 \end{aligned}$$

and f_t is continuous since g_t and h_1 are continuous. Thus $g_1 \circ h_1 \simeq h_1$. By transitivity, $g_1 \circ h_1 \simeq \mathrm{Id}_Y$, so g_1 is a homotopy inverse for h_1, so Y is homotopic to Z. It is not hard to see that Z is contractible, since it is homeomorphic to $\mathbb{R}$. Thus Y is contractible. □

Proposition 0.13 (Exercise 6b, part two). *Let Y be the subspace of $\mathbb{R}^2$ that is the union of infinitely many copies of X (see picture in Hatcher on page 18). The Y does not deformation retract onto any point.*

Proof. First, suppose x is a point in the "comb" part of some copy of X. By the same argument as in part (a), there is a ball B of sufficiently small radius so that $B \cap Y$ is disconnected, and then if Y deformation retracts to x then by Exercise 5 we get a neighborhood $V \subset B \cap Y$ that is path connected, which is a contradiction.

Now suppose that z is a point in the zigzag line Z part of Y. There is a ball B of sufficiently small radius so that any sub-neighborhood of $B \cap Y$ is disconnected, because the "comb fibers" parallel to the segment where z lives are arbitrarily close to z (this still holds if z is a "corner" of Z). Then by the argument above, if Y deformation retracts onto z then there is a path connected neighborhood of z, which is a contradiction. Thus Y does not deformation retract onto any point. □

Proposition 0.14 (Exercise 6c). *Let Y be the space described in part (b) and let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$. (See picture on page 18 of Hatcher.) There is a deformation retraction in the weak sense of Y onto Z, but no true deformation retraction.*

Proof. To get a deformation retraction in the weak sense of Y onto Z, use the map $h_t : Y \to Y$ in the proof in part (b) that Y is contractible. We have $h_0 = \mathrm{Id}_Y$ and $h_(Y) \subset Z$ and $h_t(Z) \subset Z$ for all t, and we showed that $(y,t) \mapsto h_t(y)$ is continuous, so h_t is a deformation retraction in the weak sense of Y onto Z.

Now we show that there is no deformation retraction of Y onto Z. First, note that since Z is homeomorphic to $\mathbb{R}$, Z does deformation retract onto a point. If there were a deformation retraction of Y onto Z, then that retraction followed by a deformation retraction of Z to a point would give a deformation retraction of Y to a point, which is impossible by part (b). □

As usual, in the next few propositions, the word "map" means continuous function. Keep in mind that the composition of continuous functions is continuous.

Proposition 0.15 (Exercise 10, part one). *A space X is contractible if and only if for every space Y and every map $f : X \to Y$, f is nullhomotopic.*

Proof. Suppose that for every space Y, every map $f : X \to Y$ is nullhomotopic. In particular, we can choose $Y = X$, and $f = \mathrm{Id}_X$. By hypothesis, f is nullhomotopic, so X is contractible.

Now suppose that X is contractible. Let Y be a space and $f : X \to Y$. Let $h_t : X \to X$ be a homotopy with $h_0 = \mathrm{Id}_X$ and $h_1(x) = x_0$. Then $\widetilde{h}_t = f h_t : X \to Y$ satsifies $\widetilde{h}_0 = f\,\mathrm{Id}_X = f$ and $\widetilde{h}_1(x) = f h_1(x) = f(x_0)$, which is constant. Thus $\widetilde{h}_t$ is a homotopy from f to a constant map, so f is nullhomotopic. □

Proposition 0.16 (Exercise 10, part one). *A space X is contractible if and only if for every space Y and every map $f : Y \to X$, f is nullhomotopic.*

Proof. Suppose that for every space Y, every map $f : Y \to X$ is nullhomotopic. In particular, we can choose $Y = X$ and $f = \mathrm{Id}_X$, so Id_X is nullhomotopic, so X is contractible.

Now suppose that X is contractible. Let Y be a space and $f : Y \to X$ a map. Let $h_t : X \to X$ be a homotopy with $h_0 = \mathrm{Id}_X$ and $h_1(x) = x_0$. Then $\widetilde{h}_t = h_t f : Y \to X$ satsifies $\widetilde{h}_0 = \mathrm{Id}_X f = f$ and $\widetilde{h}_1(x) = h_1 f(x) = x_0$, which is constant. Thus $\widetilde{h}_t$ is a homotopy from f to a constant map, so f is nullhomotopic. □

The next lemma is quite trivial, but removes any doubt.

Lemma 0.17 (for Exercise 13). *Let $f : X \to Y$ and $g : Y \to Z$ be functions, and let $A \subset X$. Then*

$$(g \circ f)|_A = g \circ (f|_A)$$

Proof. Let $\iota_A : A \hookrightarrow X$ be the inclusion. Then $f|_A = f \circ \iota_A$ and $(g \circ f)|_A = (g \circ f) \circ \iota_A$, so by associtivity of function composition,

$$(g \circ f)|_A = (g \circ f) \circ \iota_A = g \circ (f \circ \iota_A) = g \circ (f|_A)$$

□

Proposition 0.18 (Exercise 13). *Let X be a space and $A \subset X$, and suppose r_t^0 and r_t^1 are deformation retractions of X onto A. Then there is a "continuous family" r_t^s such that r_t^s is a deformation retraction of X onto A for each $s \in [0,1]$. The family is continuous in the sense that the map $X \times I \times I \to X$ given by $(x,s,t) \mapsto r_t^s(x)$ is continuous.*

Proof. (Note: This proof is somewhere between incomplete and useless. Nevertheless, this line of thinking seems profitable.) Because r_t^0 and r_t^1 are deformation retractions, we have

$$r_0^0 = r_0^1 = \mathrm{Id}_X \qquad r_t^0|_A = r_t^1|_A = \mathrm{Id}_A \qquad r_1^0(X) = r_1^1(X) = A$$

For $s \in [0,1]$, define $h_t^s = r_{t(1-s)}^0 \circ r_{ts}^1$. This is well-defined because for $s,t \in [0,1]$, we have $t(1-s), ts \in [0,1]$. Then we check that

$$\begin{aligned} h_t^0 &= r_t^0 \circ r_0^1 = r_t^0 \circ \mathrm{Id}_X = r_t^0 \\ h_t^1 &= r_0^0 \circ r_t^1 = \mathrm{Id}_X \circ r_t^1 = r_t^1 \end{aligned}$$

So we can take $r_t^s = h_t^s$. Both of the following are compositions of continuous functions,

$$\begin{aligned} &(x,s,t) \mapsto (x,ts) \mapsto r_{ts}^1(x) \\ &(x,s,t) \mapsto (x,t(1-s)) \mapsto r_{t(1-s)}^0(x) \end{aligned}$$

so the composition

$$(x,s,t) \mapsto r_{t(1-s)}^0 \circ r_{ts}^1(x)$$

is also continuous. We check that each r_s^t is a deformation retraction onto A.

$$\begin{aligned} r_0^s &= r_0^0 \circ r_0^1 = \mathrm{Id}_X \circ \mathrm{Id}_X = \mathrm{Id}_X \\ r_t^s|_A &= (r_{t(1-s)}^0 \circ r_{ts}^1)|_A = r_{t(1-s)}^0 \circ \left(r_{ts}^1|_A\right) = r_{t(1-s)}^0 \circ \mathrm{Id}_A = \mathrm{Id}_A \circ \mathrm{Id}_A = \mathrm{Id}_A \end{aligned}$$

However, we do not know that $r_1^s(X) = A$, so we can't say that r_t^s is a family of deformation retractions. □